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LETTER TO THE EDITOR

Some further results on a kinetic critical phenomenon

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Abstract. A critical cluster growth phenomenon is studied in which the number of particles is locally conserved modulo 2. This puts it in a new universaltiy class, although the process is superficially similar to directed percolation. We give much improved values for the critical exponents in one dimension, and we present first attempts towards a field theoretic description.

Some time ago, Krause, von der Twer and the present author [1] studied a kinetic critical phenomenon which obviously was in a new universality class which had not been studied before. This was somewhat surprising as the model seemed very natural. It consists of a multiparticle system, with particles diffusing, annihilating, and reproducing autocatalytically. The critical phenomenon occurs when the reproduction rate is just large enough to let a cluster of particles grow beyond all limits. This is of course very similar to the contact process [2] (or 'Schlögl model' [3]) interpretation of directed percolation [4]. The difference with directed percolation is that we assume the number of particles to be conserved modulo 2, i.e. the elementary reactions are

$$X \to 3X \qquad \text{reproduction, rate } \mu \tag{1a}$$

$$2X \to 0 \qquad \text{annihilation, rate } \lambda \tag{1b}$$

while in the contact process we have $X \to 2X$ instead of (1a).

This conservation modulo 2 of the particle number might not seem very natural, but in one dimension it becomes much more natural if we view the objects 'X' not as particles but as kinks (or 'Bloch walls') in a spin system. Indeed, the actual realisation of (1) was—and will be in the present paper—via a probabilistic cellular automaton, with the rules

$$\begin{aligned} &000, 010, 111 \to 0 \qquad 001, 100, 101 \to 1 \\ &011, 100 \to \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases} \end{aligned} \tag{2}$$

The evolution of a random pattern under this rule is such that for small p a regular chessboard pattern emerges, while the pattern stays random for large p. To demonstrate this, we show in figure 1 typical spacetime patterns created by (2). In these figures, we have plotted a white pixel if the spin was different from its left neighbour spin, and we have plotted a black pixel if both spins were the same. Initial configurations consisted of alternating spins except for a single kink, where two neighbouring spins coincided. We see that for large p ($p \gtrsim p_c \approx 0.54$) the kink widens to form an infinite cluster of kinks, while it stays narrow for $p < p_c$. Notice that the pattern cannot become completely ordered for $p < p_c$ since we have started with an odd number of kinks.

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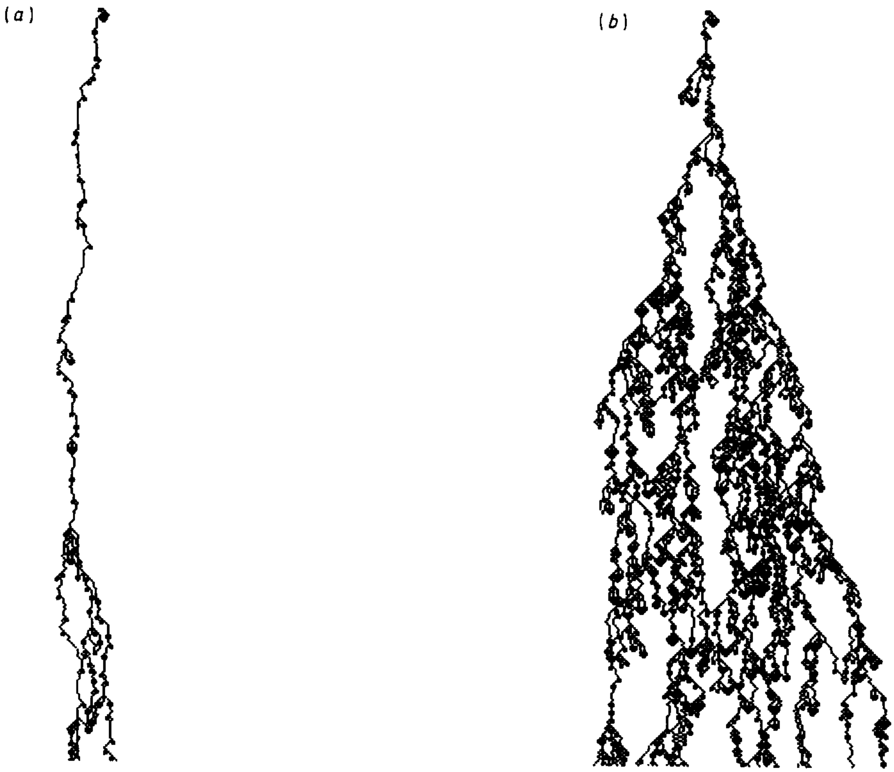

Figure 1. Clusters generated with (2) and with (*a*) $p = 0.45$ and (*b*) $p = 0.63$. In both cases, the initial configuration consisted of alternating spins except for a single kink, and pixels are drawn in black if two neighbouring spins coincide.

In [1], equation (2) was called 'model B' and the same qualitative behaviour was found in another model (called 'A' in [1]). In the present letter, we only present results for (2). We have, however, also performed additional simulations with model A, and verified that both seem to be in the same universality class.

In higher dimensions, the interpretation of the particles X as kinks breaks down, but we can of course also study (1) in its own right without this interpretation.

If (1) represents a critical phenomenon, the first question concerns the mean-field approximation. From (1) we obtain the rate equation

$$\frac{\mathrm{d}n}{\mathrm{d}t} = 2\mu n - 2\lambda n^2. \tag{3}$$

It describes the evolution of the density n when fluctuations can be neglected, which is true if the space is infinite dimensional or if diffusion is infinitely fast. Notice that (3) is the same as for the contact process with reproduction rate 2μ. That is, in the rate approximation we cannot distinguish between (1) and directed percolation.

Let us next discuss the upper critical dimension, i.e. the dimension above which (3) gives a qualitatively correct description. We claim that it is $d_c = 4$, again just as for directed percolation.

To prove this, we have first to give a field theoretic formulation. For interacting particle systems, such a formulation was given in [5, 6]. In this formalism, one has a particle annihilation field $\psi(x)$ and a creation field $\pi(x)$, with the usual commutation relations $[\psi(x), \pi(y)] = \delta(x-y)$. For extended systems, the 'inclusive' formalism of [6] is preferable, in which π is related to the Hermitian conjugate of ψ via $\pi(x) = 1+\psi^{\dagger}(x)$. Denoting the diffusion coefficient by D, we then straightforwardly obtain [6] the Liouvillean density

$$L = -D\nabla\psi^{\dagger}\nabla\psi + \mu(2\psi^{\dagger} + 3\psi^{\dagger 2} + \psi^{\dagger 3})\psi - \lambda(2\psi^{\dagger} + \psi^{\dagger 2})\psi^2. \tag{4}$$

A standard dimensional analysis [7] shows that near $d = 4$ the only terms relevant in a perturbative treatment are

$$L_{\mathrm{eff}} = -D\nabla\psi^{\dagger}\nabla\psi + 2\mu\psi^{\dagger}\psi + 3\mu\psi^{\dagger 2}\psi - 2\lambda\psi^{\dagger}\psi^2. \tag{5}$$

This is, however, exactly the same as the effective Liouvillean for the critical behaviour of directed percolation [7].

This argument not only says that both models have the same d_c. Indeed, it also suggests that the exponents are the same near d_c. In general, critical exponents are assumed to be analytic functions of the dimension. If this were also true for the present model, we would be led to the conclusion that (1) is in the same universality class as directed percolation.

The problem for the theoretical treatment is that there seems to exist no way of taking into account the constraint that the number of X is to be conserved modulo 2. To be sure, this is guaranteed by (4), but the terms in (4) responsible for it are all perturbatively irrelevant, and are thus neglected in the standard ε expansion. A similar case where the ε expansion gives a wrong result for a critical phenomenon since important terms are perturbatively irrelevant is discussed in [7].

It is quite obvious that the critical phenomenon described by (1) cannot be in the same universality class as directed percolation. In the latter, a single-particle seed leads to a cluster which at the critical point dies according to a power law with a non-zero universal exponent. In the present case, such a cluster cannot die (see figure 1), and hence both phenomena must be in different universality classes.

While in [1] we had performed simulations starting from a disordered initial state, we present in the following results of simulations of (2) with single-kink initial states. For the clusters generated in this way we expect the following scaling law for the density $\rho(x, t, \varepsilon)$ of the X (i.e. of kinks)

$$\rho(x, t, \varepsilon) \cong t^{-\alpha}\varphi(\varepsilon x^{1/\nu_{\perp}}, \varepsilon t^{1/\nu_{\parallel}}) \tag{6}$$

where $\varepsilon = p - p_c$ and where $\varphi(\xi, \tau)$ is analytic near $\xi = 0$ and $\tau = 0$. For the RMS size of the cluster in x and for its maximal size, this gives at the critical point $\varepsilon = 0$

$$\langle(x - \langle x(t)\rangle)^2\rangle^{1/2} \sim \langle x_{\max}(t) - x_{\min}(t)\rangle \sim t^{z/2} \tag{7}$$

with $z = 2\nu_{\perp}/\nu_{\parallel}$. For the growth of the average particle (or, rather, kink) number, it gives similarly

$$N(t) \sim t^{\eta}\Psi(\varepsilon t^{1/\nu_{\parallel}}) \tag{8}$$

with $\eta = \nu_{\perp}/\nu_{\parallel} - \alpha$, which at the critical point gives simply

$$N(t) \sim t^{\eta}. \tag{9}$$

Other exponents related to this ansatz are the exponent β describing the growth of the average density in the stationary supercritical state:

$$\rho(\varepsilon) = \lim_{t\to\infty} \rho(x, t, \varepsilon) \sim \varepsilon^{\beta} \qquad \varepsilon > 0 \tag{10}$$

with $\beta = \nu_{\parallel}\alpha = \nu_{\perp} - \nu_{\parallel}\eta$, and the fractal dimension of the cluster $d_F = d - \beta/\nu_{\perp}$. The decay of the density from a disordered state at the critical point should finally be $\rho(t) \sim t^{-\alpha}$.

In order to test this scaling ansatz, we have simulated for each value of p about 10^4 clusters, with times up to $t \approx 27\,000$. These simulations represent a sample more than two orders of magnitude larger than that of [1]. While the simulations in [1] had been done on a mainframe, the present ones resulted from about 6 weeks of CPU time on an ATARI MEGA ST home computer.

In figure 2 we show $N(t)$ as a function of t on a log-log plot, while $\langle x_{max} - x_{min} \rangle$ is shown in figure 3. We see indeed straight lines for large t, for $p \approx 0.54$. A more

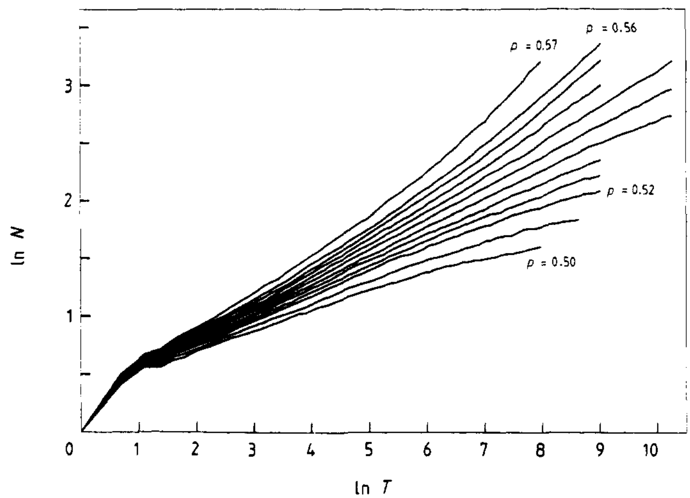

Figure 2. Average particle number N as a function of time t, on doubly logarithmic plot. The curves present—in growing order—the values $p = 0.5$, 0.51, 0.52, 0.525, 0.53, 0.535, 0.54, 0.545, 0.55, 0.555, 0.56 and 0.57.

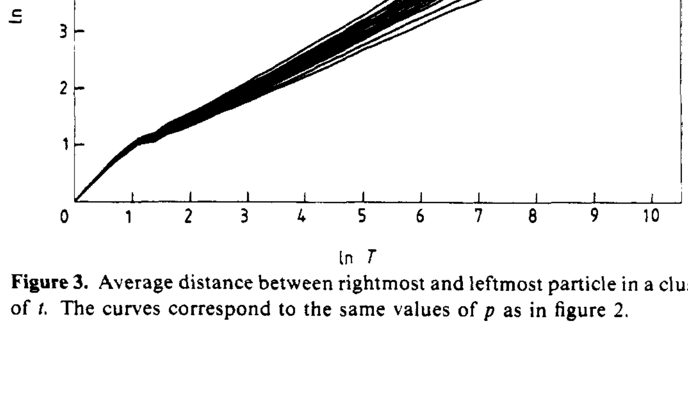

Figure 3. Average distance between rightmost and leftmost particle in a cluster as a function of t. The curves correspond to the same values of p as in figure 2.

careful analysis gives

$$p_c = 0.5403 \pm 0.0013 \tag{11}$$

$$\eta = 0.272 \pm 0.012 \tag{12}$$

$$z = 1.11 \pm 0.02. \tag{13}$$

Fitting the dependence of $N(t)$ on p with (8), we finally obtain

$$\nu_{\parallel} = 3.3 \pm 0.2. \tag{14}$$

From these, we then get $\alpha = 0.283 \pm 0.16$ and $\beta = 0.94 \pm 0.06$.

The values for p_c and for the critical exponent α are in agreement with the values of [1]. The value of β was underestimated in [1]; its present value is compatible with $\beta = 1$. Values similar to (11)-(14) were obtained from (less extensive) simulations with model A.

We conclude that we have confirmed the existence of a novel critical phenomenon as proposed in [1], with considerably higher statistics. We have shown that the critical dimension is $d = 4$, just as in directed percolation. We have also shown that the standard analytic method (the ε expansion) would predict the wrong result that both phenomena—directed percolation and the present model—are in the same universality class. To find the correct analytic treatment which can keep track of the difference between the two models is an open challenge.

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